

# From nesting to dressing

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## Abstract

In integrable field theories the S-matrix is usually a product of a relatively simple matrix and a complicated scalar factor. We make an observation that in many relativistic integrable field theories the scalar factor can be expressed as a convolution of simple kernels appearing in the nested levels of the nested Bethe ansatz. We formulate a proposal, up to some discrete ambiguities, how to reconstruct the scalar factor from the nested Bethe equations and check it for several relativistic integrable field theories. We then apply this proposal to the AdS asymptotic Bethe ansatz and recover the dressing factor in the integral representation of Dorey, Hofman and Maldacena.

## 1 Introduction

The AdS/CFT correspondence [1], which links  $\mathcal{N} = 4$  Super Yang-Mills theory and  $AdS_5 \times S^5$  superstring theory provides a fascinating and powerful

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method for studying the behaviour of gauge theory at strong coupling. Yet that same feature makes it very difficult to test in the whole range of coupling constants. If we want to make contact with perturbative gauge theory we are forced to investigate the deeply quantum regime of string theory. On the other hand if we start from gauge theoretical constructions, passage to strong coupling is very difficult.

The discovery of integrability [2, 3, 4, 5, 6, 7] on both sides of the correspondence gave new tools for making the strong-weak coupling interpolation. On the gauge theory side integrable spin chains have been investigated for general operators at 1-loop [8]. The integrable structures on both sides of the correspondence were analyzed from the point of view of finite-gap solutions [9]. Then the spin chain formulation was extended to an all-loop asymptotic Bethe ansatz [10, 11, 12]. Yet it turned that in order to describe correctly strong coupling string physics one has to introduce a scalar function – the so-called ‘dressing factor’ [13, 14, 15] which was unconstrained by integrability.

In parallel, the S-matrix (which could be interpreted both as an ingredient of the spin chain language or as a worldsheet S-matrix for superstring excitations) was found to be determined by symmetry again up to the scalar factor [16]. Despite the lack of relativistic invariance, constraints from crossing symmetry were derived [17], and a dressing factor(s) solving these constraints was found [18]. Later in [19] a specific member of the class of solutions was chosen.

One of the intriguing features of the form of the dressing factor was its extreme complexity. Even though integral representations have been found [20, 21, 22], the complicated formulas prompted some conjectures that the dressing factor arises dynamically from some hidden levels of a more fundamental Bethe ansatz [23]. This idea was earlier partly realised in some models [24, 25, 26], and other works [27] suggested a construction which works for large length. However no completely satisfactory construction along these lines has been found so far.

The aim of this paper is to show that the complexity of the AdS dressing factor is only apparent and that it shares a lot of its structure with quite simple ordinary relativistic integrable field theories. We make an observation that the nested structure of Bethe ansatz is intertwined with the form of the scalar factor and we found a ‘phenomenological’ procedure (up to a set of discrete ambiguities) of constructing the dressing factor out of simple ingredients entering the nested levels. The outcome is a generic structure

of the scalar factor as (multiple) convolution of simple kernels appearing in the nested Bethe equations very much reminiscent of the Dorey, Hofman, Maldacena integral representation of the AdS dressing factor.

The plan of this paper is as follows. In section 2 we review the basic facts about the BHL/BES dressing factor and quote the integral expression of [22]. In section 3 we describe thermodynamic Bethe ansatz (TBA) for relativistic integrable field theories and analyze the case of  $(RSOS)_3$  model which is the basis for our observation. Then we summarize our procedure and check that it works for the  $O(4)$  model and more generally for all  $O(2n)$  models. Then in section 4 we move to the case of  $AdS_5 \times S^5$  and show how one can recover the integral formula of Dorey, Hofman and Maldacena from our construction which has as its input just the asymptotic Bethe ansatz. We close the paper with a conclusion and two appendices.

## 2 The BHL/BES dressing factor

The dressing factor of the  $AdS_5 \times S^5$  superstring worldsheet theory is currently believed to be given by the BHL/BES expression [18, 19]. This form satisfies all the known constraints both at weak and at strong coupling as well as it ensures that the S-matrix has crossing symmetry [17]. Its general form is given by the factorization

$$\sigma^2(x_q, x_p) = \frac{R^2(x_q^-, x_p^-) R^2(x_q^+, x_p^+)}{R^2(x_q^-, x_p^+) R^2(x_q^+, x_p^-)} \quad (1)$$

where each of the factors is given by

$$R(x, y) = e^{i\chi(x, y)} \quad (2)$$

where  $\chi(x, y) = \tilde{\chi}(x, y) - \tilde{\chi}(y, x)$  is antisymmetric and defined through the series expansion [18, 19]

$$\tilde{\chi}(x, y) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} \frac{-c_{r,s}(g)}{(r-1)(s-1)} \frac{1}{x^{r-1} y^{s-1}} \quad (3)$$

where  $c_{r,s}(g)$  have the strong coupling expansion

$$c_{r,s}(g) = \sum_{n=0}^{\infty} c_{r,s}^{(n)} \frac{1}{g^{n-1}} \quad (4)$$

with the coefficients  $c_{r,s}^{(n)}$  given by

$$c_{r,s}^{(0)} = \delta_{r+1,s} \quad (5)$$

for the leading AFS part [13],

$$c_{r,s}^{(1)} = \frac{(-1)^{r+s} - 1}{\pi} \cdot \frac{(r-1)(s-1)}{(r+s-2)(s-r)} \quad (6)$$

for the 1-loop HL part<sup>1</sup> [15] and the BHL/BES choice [18, 19]

$$c_{r,s}^{(n)} = \frac{(1 - (-1)^{r+s})\zeta(n)}{2(-2\pi)^n \Gamma(n-1)} (r-1)(s-1) \frac{\Gamma(\frac{1}{2}(s+r+n-3))\Gamma(\frac{1}{2}(s-r+n-1))}{\Gamma(\frac{1}{2}(s+r-n+1))\Gamma(\frac{1}{2}(s-r-n+3))} \quad (7)$$

for  $n \geq 2$

The above strong coupling expansion is only asymptotic. In [19] there appeared a corresponding expansion at weak coupling

$$c_{r,s}(g) = \sum_{n=0}^{\infty} \tilde{c}_{r,s}^{(-n)} g^{1+n} \quad (8)$$

which could be evaluated to give

$$c_{r,s}(g) = 2 \cos \left( \frac{1}{2} \pi (s-r-1) \right) \cdot (r-1)(s-1) \int_0^{\infty} dt \frac{J_{r-1}(2gt) J_{s-1}(2gt)}{t(e^t - 1)} \quad (9)$$

which agrees with the asymptotic strong coupling formulas quoted above [28]. Later a convenient integral expression based on (9) was derived by Dorey, Hofman and Maldacena [22]

$$\begin{aligned} \tilde{\chi}_{DHM}(x, y) &= -i \int_{C_1} \frac{dz_1}{2\pi} \int_{C_2} \frac{dz_2}{2\pi} \frac{1}{x - z_1} \frac{1}{y - z_2} \cdot \\ &\quad \cdot \log \Gamma \left( 1 + ig \left( z_1 + \frac{1}{z_1} - z_2 - \frac{1}{z_2} \right) \right) \end{aligned} \quad (10)$$

which resums all terms in the weak coupling expansion<sup>2</sup>. In the above expression the integration contours  $C_i$  are unit circles. Even though the above

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<sup>1</sup>Based in particular on the 1-loop computations in [14].

<sup>2</sup>Other forms of integral expressions for the dressing kernel have been derived earlier in [20, 21], but the one from [22] will be most convenient for our purposes.

expression is much simpler than the rather complicated expressions for the dressing factor coefficients it is still rather formidable. One of the main points of this paper is to show that the above structure is in fact quite natural. In the next section we will return to conventional relativistic integrable field theories and show that the scalar factors for a wide range of theories with *nondiagonal* scattering have a very similar form.

### 3 Integrable relativistic theories – TBA

The point of departure of our observation was the rather surprising simplicity of thermodynamic Bethe equations for the simplest relativistic integrable theory with nondiagonal scattering: the  $(RSOS)_3$  model. Before we describe it in detail let us briefly review the thermodynamic Bethe ansatz (TBA) in the case of diagonal scattering and a single species of particles [29].

The aim of the thermodynamic Bethe ansatz in its original form is to compute the ground state energy  $E_0(L)$  for an integrable quantum field defined on a cylinder of circumference  $L$ . The ground state energy can be reconstructed from the euclidean partition function

$$E_0 = - \lim_{R \rightarrow \infty} \frac{1}{R} \log Z(R, L) \quad (11)$$

The direct computation of this partition function in an interacting, even integrable, theory is very complicated due to virtual particles going around the cylinder and interacting with each other. The Thermodynamic Bethe Ansatz amounts to computing the same partition function treating  $R$  as space and  $L$  as a compactified time, i.e. inverse temperature. Now since the space coordinate is decompactified the complications due to virtual particles go away and one can use ordinary Bethe ansatz quantization:

$$e^{imR \sinh \theta_i} \prod_{j \neq i} S(\theta_i - \theta_j) = 1 \quad (12)$$

which, after taking logarithms, becomes

$$mR \sinh \theta_i + \sum_{j \neq i} \delta(\theta_i - \theta_j) = 2\pi n_i \quad (13)$$

where  $n_i$  is an integer and  $\delta = -i \log S$ . If we look at a specific solution to the system (13), we find that it corresponds to a set of, non-necessarily

consecutive, integers. We will call those roots ‘occupied roots’. Then because  $R \rightarrow \infty$ , we are led to consider continuous distributions of roots. Taking the derivative w.r.t.  $\theta_i$  we thus obtain

$$\rho(\theta) + \rho^h(\theta) = \frac{1}{2\pi} m R \cosh \theta + \int \phi(\theta - \theta') \rho(\theta') d\theta' \quad (14)$$

where  $\rho(\theta)$  is the density of occupied roots,  $\rho^h(\theta)$  is the density of *unoccupied* roots (‘holes’) while  $\phi(\theta)$  is the S-matrix kernel

$$\phi = \frac{1}{2\pi i} \partial_\theta \log S \quad (15)$$

At this stage, equation (14) is a relation between two unknown quantities – the densities of particles and holes. In the further steps of the TBA procedure one minimizes the free energy obtaining a second equation which links  $\rho(\theta)$  to  $\rho^h(\theta)$  thus forming, together with (14), a closed system of equations which determines  $\rho$  and  $\rho^h$  and consequently the partition function and the ground state energy  $E_0(L)$ .

For our applications, it is already a feature of the equation (14) written with  $\rho$  and  $\rho^h$  taken to be independent which will be at the basis of our proposal. Nothing will depend on the further steps of TBA so we will not describe them here in more detail.

### 3.1 The $(RSOS)_3$ model

In this section we would like to consider, following [30], the analogues of the equation (14) for one of the simplest theories with nondiagonal scattering – the  $(RSOS)_3$  model. Since the S-matrix is now nondiagonal, instead of the ordinary Bethe ansatz one has to deal with additional levels of nesting and introduce additional species of roots which do not carry any momentum or energy but which encode the diagonalization of the transfer matrix.

The S-matrix of the RSOS model has the following structure:

$$\hat{S} = \frac{1}{\cosh^{\frac{1}{2}} \theta} \cdot \underbrace{e^{\frac{i}{4} \int_0^\infty \frac{dt}{t} \frac{\sin \frac{t\theta}{2}}{\cosh^2 \frac{t}{2}}}}_{\sigma(\theta)} \cdot \begin{pmatrix} \text{Trigonometric} \\ \text{functions of } \theta \end{pmatrix} \quad (16)$$

We see here that the scalar factor  $\sigma(\theta)$  necessary for imposing crossing invariance is much more complicated than the trigonometric functions appearing in

the matrix structure. This is a generic situation in theories with nondiagonal scattering.

When applying Bethe ansatz quantization to the S-matrix (16) we have one additional flavour of roots:  $x_i$  which come in two varieties

$$x_i^\pm = y_i \pm i\frac{\pi}{2} \quad (17)$$

The  $y_i$ 's satisfy the equation

$$\prod_{k=1}^N \frac{\sinh\left(\frac{y_i - \theta_k}{2} + i\frac{\pi}{4}\right)}{\sinh\left(\frac{y_i - \theta_k}{2} - i\frac{\pi}{4}\right)} = \pm 1 \quad (18)$$

where the corresponding imaginary parts of  $x_i$ 's may be either  $+i\pi/2$  or  $-i\pi/2$ . The ‘physical’ momentum carrying roots satisfy the equation

$$e^{imR \sinh \theta_k} \prod_l \sigma(\theta_k - \theta_l) \prod_i \sinh \frac{\theta_k - x_i}{2} \cdot (\dots) = 1 \quad (19)$$

where  $(\dots)$  are terms which do not contribute in the thermodynamic limit. Note the appearance of the complicated ‘dressing phase’  $\sigma(\theta)$ . In [30], Zamolodchikov derived the continuous equations of this model in the course of applying the TBA procedure. The first equation is

$$\rho^+(y) + \rho^-(y) = \frac{1}{2\pi} \int \frac{1}{\cosh(y - \theta)} \rho(\theta) d\theta \equiv K * \rho \quad (20)$$

which involves the rather simple kernel  $K = \frac{1}{2\pi} \frac{1}{\cosh(y - \theta)}$  coming from the nested level equation. The second equation quoted by Zamolodchikov reads

$$\begin{aligned} \rho(\theta) + \rho^h(\theta) &= \frac{1}{2\pi} mR \cosh \theta + \frac{1}{2\pi} \int \frac{1}{\cosh(\theta - y)} \rho^+(y) dy \\ &\equiv \frac{1}{2\pi} mR \cosh \theta + K * \rho^+ \end{aligned} \quad (21)$$

Let us emphasize that this equation looks extremely surprising. It came from taking the continuum limit of (19) and so should incorporate somehow the dressing kernel ( $\phi_\sigma \equiv \frac{1}{2\pi i} \partial_\theta \log \sigma$ ). Yet this equation only involves the much simpler ‘nesting kernel’  $K$ .

In order to see in detail how this came about let us write (the imaginary part<sup>3</sup> of) the continuum version of (19):

$$\rho + \rho^h = \frac{1}{2\pi} mR \cosh \theta + \phi_\sigma * \rho + \frac{1}{4\pi} \int \frac{\rho^+(y) - \rho^-(y)}{\cosh(\theta - y)} dy \quad (22)$$

Now we may express  $\rho^-(y)$  from (20) and plug it into the above equation. We obtain

$$\rho + \rho^h = \frac{1}{2\pi} mR \cosh \theta + \phi_\sigma * \rho - \frac{1}{2} K * K * \rho + K * \rho^+ \quad (23)$$

This is indeed equal to (21) once we note that the specific dressing phase of the RSOS model satisfies

$$\phi_\sigma = \frac{1}{2} K * K \quad (24)$$

This surprising expression is the main motivation of our work. It gives an expression for the dressing kernel, which is a complicated looking transcendental function, as a convolution of much simpler kernels which come from nested levels of Bethe equations. The appearance of such a formula suggests that the complicated form of the dressing phases (S-matrix scalar factors) may just be a byproduct of some simple structure extracted from the ‘simple’ matrix part of the S-matrix.

## 3.2 The proposal

The surprising cancelation observed in the case of TBA equations for the  $(RSOS)_3$  model prompted us to conjecture that such a phenomenon may be more general. If we abstract the basic steps leading to the cancelation the following procedure suggests itself.

First consider the momentum carrying equation in the Bethe ansatz and write it in the continuum limit as for TBA. In the case of nondiagonal scattering this equation will involve auxiliary Bethe roots involved in the diagonalization of the transfer matrix:

$$\dots + \sum_i K_i * \rho_i \quad (25)$$

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<sup>3</sup>The real part is also satisfied – see [30].



where  $\rho_i$  are the densities of the auxiliary roots. Then we use the nested levels of the Bethe equations to express the  $\rho_i$ 's in terms of the density of the momentum carrying roots  $\rho$ :

$$\rho_i = \tilde{K}_i * \rho + \dots \quad (26)$$

where  $\dots$  may stand for e.g. densities of holes etc. Then the observation made for the  $(RSOS)_3$  model suggests that the dressing kernel has an expression of the form

$$K_\sigma = - \sum_i K_i * \tilde{K}_i \quad (27)$$

The origin of this ‘phenomenological’ observation is unclear to us. We suspect that it must be a feature of some internal consistency between the structural properties of the nested Bethe ansatz and crossing which fixes the overall scalar factor of the S-matrix which we call here generically ‘the dressing factor’. Note that there is some ambiguity in the above construction. The nested equations could be inverted which would lead to various sign changes in the kernels (which arise through taking the logarithms of the equations – this is especially evident in the case of  $O(2n)$  models to be discussed below). We do not know what fixes the correct signs – what is surprising for us is that such a choice *exists* at all. In order to check whether this is not just a coincidence, we will analyse in the next section the  $O(4)$  model from the same perspective and then we will verify that the same procedure works also for all  $O(2n)$  models.

Then in section 4, we will proceed to use the same mechanism for the  $AdS_5 \times S^5$  superstring/ $\mathcal{N} = 4$  SYM Bethe ansatz.

### 3.3 $O(4)$ model

The  $O(4)$  model is described by the S-matrix

$$\hat{S}_{ab}^{cd}(\theta) = \sigma^2(\theta) \frac{\theta}{\theta - i} \left[ \delta_a^c \delta_b^d - \frac{i}{\theta} \delta_a^d \delta_b^c - \frac{i}{i - \theta} \delta_{ab} \delta^{cd} \right] \quad (28)$$

where the scalar factor has the integral representation

$$\sigma^2(\theta) = e^{2i \int_0^\infty \frac{dk}{k} \sin k\theta \cdot \frac{2e^{-k}}{1+e^{-k}}} \quad (29)$$

alternatively it can be expressed as a product of Gamma functions [31].

Let us quote the Bethe equations of the  $O(4)$  model. Apart from the physical  $\theta_k$ 's we have two sets of auxillary roots:  $u_i$ 's and  $v_j$ 's. There are three Bethe equations:

$$1 = \prod_k \frac{u_j - \theta_k - \frac{i}{2}}{u_j - \theta_k + \frac{i}{2}} \prod_{i \neq j} \frac{u_j - u_i + i}{u_j - u_i - i} \quad (30)$$

$$e^{-im \sinh \pi \theta_k} = \prod_{j \neq k} \sigma^2(\theta_k - \theta_j) \prod_l \frac{\theta_k - u_l + \frac{i}{2}}{\theta_k - u_l - \frac{i}{2}} \prod_l \frac{\theta_k - v_l + \frac{i}{2}}{\theta_k - v_l - \frac{i}{2}} \quad (31)$$

$$1 = \prod_k \frac{v_j - \theta_k - \frac{i}{2}}{v_j - \theta_k + \frac{i}{2}} \prod_{i \neq j} \frac{v_j - v_i + i}{v_j - v_i - i} \quad (32)$$

Let us now define the continuum limit of these equations which in principle would be the starting point of TBA<sup>4</sup>. It is convenient to use the notation

$$\phi_a(x) \equiv \frac{1}{2\pi i} \partial_x \log\left(\frac{x + ia}{x - ia}\right) = -\frac{1}{\pi} \frac{a}{x^2 + a^2} \quad (33)$$

$$\phi_\sigma(\theta) \equiv \frac{1}{2\pi i} \partial_\theta \log(\sigma(\theta)) \quad (34)$$

We obtain

$$\rho_u + \rho_u^h = \phi_{-\frac{1}{2}} * \rho + \phi_1 * \rho_u \quad (35)$$

$$\rho + \rho^h = \frac{1}{2\pi} mR \cosh \theta + 2\phi_\sigma * \rho + \phi_{\frac{1}{2}} * \rho_u + \phi_{\frac{1}{2}} * \rho_v \quad (36)$$

$$\rho_v + \rho_v^h = \phi_{-\frac{1}{2}} * \rho + \phi_1 * \rho_v \quad (37)$$

Now let us perform similar manipulations as in the  $(RSOS)_3$  model. First let us express the auxillary densities  $\rho_u$  and  $\rho_v$  in terms of  $\rho$ . We obtain

$$\rho_{u,v} = (1 - \phi_1)^{-1} * \phi_{-\frac{1}{2}} * \rho + \dots \quad (38)$$

where the dotted terms  $\dots$  contain only hole densities. Then according to our proposal we would expect the dressing kernel to be given by the formula

$$\phi_\sigma = -\phi_{\frac{1}{2}} * (1 - \phi_1)^{-1} * \phi_{-\frac{1}{2}} \quad (39)$$

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<sup>4</sup>Note however that the TBA equations for  $O(4)$  models are more complicated [32, 33].

This convolution can be easily evaluated using Fourier transforms and checked that the result indeed exactly coincides with (29)! So we see that again the dressing kernel arises as convolution of simpler nested kernels which come from the nested levels of the Bethe equations. In fact the above decomposition is structurally reminiscent of the DHM formula (10) – it is also given by a double convolution with the extreme factors being quite simple.

Since we do not have a real understanding why the above procedure seems to work, we decided to check it also in the more general case of  $O(2n)$  models - as the  $O(4)$  model is really quite special being essentially a product of two  $SU(2)$  S-matrices.

### 3.4 $O(2n)$ models

For  $n \geq 2$  the  $O(2n)$   $\sigma$ -model is described by the S-matrix which was proposed a long time ago by Zamolodchikov and Zamolodchikov [31] and takes the form

$$\hat{S}_{ab}^{cd}(\theta) = \sigma^2(\theta) \frac{\theta}{\theta - i} \left[ \delta_a^c \delta_b^d - \frac{i}{\theta} \delta_a^d \delta_b^c - \frac{i}{i(n-1) - \theta} \delta_{ab} \delta^{cd} \right] \quad (40)$$

where  $\sigma(\theta)$  is the dressing factor.

For given  $n$  we have a tower of Bethe equations, one for each type of Bethe root, which consists of the main ‘momentum carrying’ Bethe equation

$$e^{-iL \sinh(\frac{\pi}{n-1} \theta_\alpha)} = \prod_{\beta \neq \alpha}^{K_{n+1}} S_0(\theta_\alpha - \theta_\beta) \prod_{j=1}^{K_n} \frac{\theta_\alpha - u_j^{(n)} + \frac{i}{2}}{\theta_\alpha - u_j^{(n)} - \frac{i}{2}} \quad (41)$$

and  $n$  ‘nested’ equations (one for each auxillary root). This set of ‘nested’ equations can be expressed in closed form as

$$-1 = \prod_{l=1}^{n+1} \prod_{j=1}^{K_l} \frac{u_i^{(k)} - u_j^{(l)} + \frac{i}{2}(\alpha_k | \alpha_l)}{u_i^{(k)} - u_j^{(l)} - \frac{i}{2}(\alpha_k | \alpha_l)} \quad (42)$$

Here  $\{\alpha_k\}$  for  $k = 1, \dots, n$  denote the simple roots of the  $\mathfrak{so}(2n)$  Lie algebra and  $(\alpha_k | \alpha_l)$  expresses inner product defined in the root space as

$$(\alpha_k | \alpha_l) = \begin{cases} 2 & \text{for } l=k, (k=1, \dots, n) \\ -1 & \text{for } l=k+1, (k=2, \dots, n-1) \\ -1 & \text{for } k=1, l=3. \end{cases} \quad (43)$$

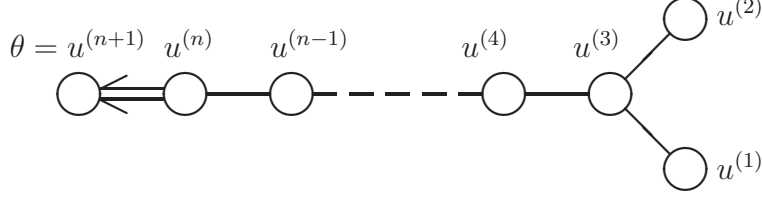


Figure 1: Dynkin diagram for the  $O(2n)$  model

Additionally, we introduced an extra root  $\alpha_{n+1}$  giving  $(\alpha_{n+1}|\alpha_k) = -\delta_{nk}$ , so that the interaction between  $\theta_\alpha$ 's and  $u_j^{(k)}$ 's are included in the common notation (42). The non-zero elements can be encoded in the Dynkin diagram (Fig.3.4).

Starting from the Bethe ansatz equations (41) and (42) we would like to take the continuum limit and transform them from the discrete to an integral form. From the form of the Dynkin diagram we see that generic nodes have only two neighbours. For these Bethe roots we obtain

$$\rho_k + \rho_k^h = \phi_{-\frac{1}{2}} * \rho_{k+1} + \phi_{-\frac{1}{2}} * \rho_{k-1} + \phi_1 * \rho_k \quad (44)$$

where  $\rho_i$  denotes the density of  $i$ -th Bethe roots and  $\rho_i^h$  the density of holes and  $\phi_a$  is defined by (33).

Apart from these nodes there are always four nodes which have different number of neighbours and thus have to be treated separately. After we take the continuum limit the Bethe equations associated with these nodes take the form

$$\rho_1 + \rho_1^h = \phi_{-\frac{1}{2}} * \rho_3 + \phi_1 * \rho_1 \quad (45)$$

$$\rho_2 + \rho_2^h = \phi_{-\frac{1}{2}} * \rho_3 + \phi_1 * \rho_2 \quad (46)$$

$$\rho_3 + \rho_3^h = \phi_{-\frac{1}{2}} * \rho_1 + \phi_{-\frac{1}{2}} * \rho_2 + \phi_{-\frac{1}{2}} * \rho_4 + \phi_1 * \rho_3 \quad (47)$$

for the auxillary roots and

$$\rho_{n+1} + \rho_{n+1}^h = \frac{1}{2(n-1)} \cosh\left(\frac{\pi}{n-1}\theta_\alpha\right) + 2\phi_\sigma * \rho_{n+1} + \phi_{\frac{1}{2}} * \rho_n \quad (48)$$

for the main, ‘momentum carrying’ roots. Again we see that this is the only equation which involves the ‘dressing factor’.

From the above equation we see, that according to our proposal, we have to express  $\rho_n$  in terms of  $\rho_{n+1}$  using (44)-(47) to get  $\rho_n = \Phi * \rho_{n+1}$  and then the dressing factor should be given by

$$\phi_\sigma = -\frac{1}{2} \phi_{\frac{1}{2}} * \Phi \quad (49)$$

The simplest way to solve the set of equations (44)-(47) is to take the Fourier transform of it. Doing it we get the set of  $n + 1$  algebraic equations which is solved in Appendix B. As a result we get

$$\hat{\phi}_\sigma = \frac{1}{2} \hat{\phi}_{-1} \frac{1 + (\hat{\phi}_{-1})^{n-2}}{1 + (\hat{\phi}_{-1})^{n-1}} \quad (50)$$

and using (34) we obtain the formula for the dressing factor

$$\sigma^2(x) = \exp \left( 2i \int_0^{+\infty} \frac{\sin(\frac{\omega\pi}{n-1}x)}{\omega} \frac{e^{-\frac{\omega\pi}{n-1}} + e^{-\omega\pi}}{1 + e^{-\omega\pi}} d\omega \right) \quad (51)$$

what agrees with the result from [33] taking into account the different conventions<sup>5</sup>.

### Ambiguities in the choice of signs

As a note of caution let us point out an ambiguity appearing in our current proposal. Let us consider e.g. the Bethe equation for the roots  $u_i^{(1)}$  and *invert* it to get

$$-1 = \prod_{l=1}^{n+1} \prod_{j=1}^{K_l} \frac{u_i^{(1)} - u_j^{(l)} - \frac{i}{2}(\alpha_1|\alpha_l)}{u_i^{(1)} - u_j^{(l)} + \frac{i}{2}(\alpha_1|\alpha_l)} \quad (52)$$

This is of course completely equivalent to the previous form. Let us now proceed to take the thermodynamical limit. The outcome will be

$$\begin{aligned} \rho_1 + \rho_1^h &= \phi_{\frac{1}{2}} * \rho_3 + \phi_{-1} * \rho_1 \\ &= -\phi_{-\frac{1}{2}} * \rho_3 - \phi_1 * \rho_1 \end{aligned} \quad (53)$$

Proceeding as before, we will obtain a different formula for the dressing factor. We do not know how to pick the correct choice on general grounds. This means that our proposal has to be supplanted by, a yet to be discovered, criterion. The nontrivial feature coming from all these computations is that nevertheless a choice exists which reproduces the correct scalar factor.

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<sup>5</sup>Rapidities have to be rescaled  $\theta \rightarrow \frac{\theta(n-1)}{\pi}$ , and  $n$  is interchanged with  $2n$ .

## 4 Dressing factor from the $AdS_5 \times S^5$ Bethe ansatz

Let us now apply our proposal to the main case of interest, namely the  $AdS$  S-matrix. As noted before, there is a (finite!) set of choices of the way one writes the equations which may give different results. In the relativistic cases of  $(RSOS)_3$  or  $O(2n)$  models there was a choice which reproduced the correct scalar factor. The same situation – a possibility of different choices – happens in the case of the asymptotic Bethe ansatz. Since we do not know a general proof of our proposal we do not know *a-priori* how to justify a particular choice. What is surprising is that a choice exists which leads to the DHM integral formula (10).

Let us start with the Beisert-Staudacher asymptotic Bethe ansatz written in the form of 7 equations with the grading corresponding to the  $sl(2)$  sector. Let us first look at the momentum carrying node equation, which again is the only equation involving the dressing factor. The momentum carrying roots are  $x_4^\pm$ , which are intertwined in this equation with roots of type  $x_1, x_3, x_5$  and  $x_7$  through the terms:

$$\prod_k \frac{x_4^- - \frac{1}{x_{1,k}}}{x_4^+ - \frac{1}{x_{1,k}}} \prod_l \frac{x_4^- - x_{3,l}}{x_4^+ - x_{3,l}} \cdot (x_1 \leftrightarrow x_7, x_3 \leftrightarrow x_5) \quad (54)$$

Hence the associated kernels appearing in this equations are

$$K_{41} * \rho_1 + K_{43} * \rho_3 + (1 \leftrightarrow 7, 3 \leftrightarrow 5) \quad (55)$$

Since we have direct symmetry between the roots coming from the two sides of the Dynkin diagram from now on we will just consider the contribution of the roots  $x_{1,2,3}$ .

Now in order to use our proposal (27) we have to solve the first three equations for the densities  $\rho_1$  and  $\rho_3$  and express them in terms of  $\rho_4$  and the hole densities – of course only the part proportional to  $\rho_4$  will be relevant for us.

Let us write these three equation in the following form:

$$1 = \prod_j \frac{u_1 - u_{2,j} - \frac{i}{2g}}{u_1 - u_{2,j} + \frac{i}{2g}} \prod_k \frac{\frac{1}{x_1} - x_{4,k}^-}{\frac{1}{x_1} - x_{4,k}^+} \quad (56)$$

$$1 = \prod_j \frac{u_2 - u_{2,j} - \frac{i}{g}}{u_2 - u_{2,j} + \frac{i}{g}} \prod_l \frac{u_2 - u_{3,l} + \frac{i}{2g}}{u_2 - u_{3,l} - \frac{i}{2g}} \prod_m \frac{u_2 - u_{1,m} + \frac{i}{2g}}{u_2 - u_{1,m} - \frac{i}{2g}} \quad (57)$$

$$1 = \prod_j \frac{u_3 - u_{2,j} + \frac{i}{2g}}{u_3 - u_{2,j} - \frac{i}{2g}} \prod_k \frac{x_3 - x_{4,k}^+}{x_3 - x_{4,k}^-} \quad (58)$$

where

$$u_i \equiv x_i + \frac{1}{x_i} \quad (59)$$

We will now proceed to rewrite these equations in the continuous form:

$$\rho_1 + \rho_1^h = \phi_{-\frac{1}{2g}} * \rho_2 + K_{14} * \rho_4 \quad (60)$$

$$\rho_2 + \rho_2^h = \phi_{-\frac{1}{g}} * \rho_2 + \phi_{\frac{1}{2g}} * \rho_3 + \phi_{\frac{1}{2g}} * \rho_1 \quad (61)$$

$$\rho_3 + \rho_3^h = \phi_{\frac{1}{2g}} * \rho_2 - K_{34} * \rho_4 \quad (62)$$

Note that here the convolutions with the  $\phi_a$  kernels are defined w.r.t. the  $u_{1,2,3}$  variables. Solving first for  $\rho_2$  gives

$$\rho_2 = (1 - \phi_{-\frac{1}{g}})^{-1} * \phi_{\frac{1}{2g}} * (\rho_1 + \rho_3) + \dots \quad (63)$$

where (...) here and below stands for terms with hole densities which will not be relevant for our proposal. Plugging the above equation into the remaining ones we obtain easily

$$\rho_1 = -L * (K_{14} - K_{34}) * \rho_4 + K_{14} * \rho_4 + \dots \quad (64)$$

$$\rho_3 = L * (K_{14} - K_{34}) * \rho_4 - K_{34} * \rho_4 + \dots \quad (65)$$

where  $L$  is the kernel

$$L = \phi_{\frac{1}{2g}} * (1 - \phi_{-\frac{1}{g}})^{-1} * \phi_{\frac{1}{2g}} \quad (66)$$

Our proposal (27) then gives the following expression for the dressing kernel:

$$\phi_\sigma = -(K_{41} * K_{14} - K_{43} * K_{34} - (K_{41} - K_{43}) * L * (K_{14} - K_{34})) \quad (67)$$

In order to make this formula concrete we have to pick a choice of contours of integration for the convolutions appearing in (67).

## Contours

In the above expression the convolution is in terms of  $x$  and not in terms of  $u = x + 1/x$ , while in the nested levels we can deal with the kernel only by taking convolution w.r.t.  $u$  on the whole real line  $u \in (-\infty, \infty)$ . Hence we have to pick a corresponding contour in the  $x$  plane. There are basically two natural choices: (i) integral from  $-\infty$  to  $-1$ , then an upper or lower semicircle and then from  $1$  to  $\infty$ , or (ii) integral from  $0$  to  $-1$ , then a semicircle to  $1$  and then the interval from  $1$  to  $0$  along the positive real axis.

In order to have a resulting effective closed contour we will pick the contour (ii) for  $x_1$  and the contour (i) for  $x_3$ . Then the convolution with the kernel  $K_{41} - K_{43}$  is effectively just a convolution along *the unit circle* since

$$\int_{C_1} dx \partial_x \log \frac{x_4^- - \frac{1}{x}}{x_4^+ - \frac{1}{x}}(\dots) - \int_{C_3} dx \partial_x \log \frac{x_4^- - x}{x_4^+ - x}(\dots) = - \oint_{|x|=1} dx \partial_x \log \frac{x_4^- - x}{x_4^+ - x}(\dots) \quad (68)$$

Similarly combining the two first terms  $K_{41} * K_{14} - K_{43} * K_{34}$  together, we see that the sum becomes just an integral over the unit circle. Moreover, since for physical particles we have  $|x_4^\pm| > 1$ , these integrals *vanish*.

## Structural properties of the dressing factor

Using the above choice of contours, we thus get the following expression of the dressing kernel

$$\phi_\sigma = \frac{1}{4\pi^2} \oint_{|x|=1} dx \oint_{|y|=1} dy \partial_z \log \frac{x_4^+(z) - x}{x_4^-(z) - x} L(x, y) \partial_y \log \frac{\tilde{x}_4^+ - y}{\tilde{x}_4^- - y} \quad (69)$$

The variable  $z$  is some parameter for the momentum carrying roots which is not essential since in any case we will be integrating later w.r.t.  $z$  in order to get directly the dressing phase.

Let us note now some structural properties of the the above expression. First because of the logarithm, the terms with  $x^+$  enter (69) with opposite sign as the terms with  $x^-$ . This means that the dressing phase factorizes in exactly the expected form:

$$\sigma(x_4^\pm, \tilde{x}_4^\pm) = \frac{R(x_4^+, \tilde{x}_4^+) R(x_4^-, \tilde{x}_4^-)}{R(x_4^-, \tilde{x}_4^+) R(x_4^+, \tilde{x}_4^-)} \quad (70)$$

Another very important property of the dressing factor is its antisymmetry. A simple property of the kernel  $L(x, y)$  is that its Fourier transform is a



function of the modulus  $|k|$ . Hence the integral over  $k$  splits naturally into two parts. After a redefinition of variables and an integration by parts (see formula (72), antisymmetry follows.

Finally let us note that the kernel  $L(x, y)$ , which is obtained from the nested kernels depends on  $x$  and  $y$  only through the combination

$$x + \frac{1}{x} - y - \frac{1}{y} \quad (71)$$

exactly as in the Dorey-Hofman-Maldacena integral formula.

## The dressing factor

In order to obtain an expression directly for the dressing factor, we have to integrate w.r.t. the  $z$  parameter, and also, in order to get a symmetric expression, integrate by parts w.r.t.  $y$ . Then using the factorization (70) we get the expression for  $\chi(x_4^+, \tilde{x}_4^+)$

$$\frac{2\pi}{4\pi^2} \oint_{|x|=1} dx \oint_{|y|=1} dy \frac{1}{x_4^+ - x} \int_{-\infty}^{\infty} \frac{dk}{2\pi i k} L(k) e^{-ik \cdot (x + \frac{1}{x} - y - \frac{1}{y})} \frac{1}{\tilde{x}_4^+ - y} \quad (72)$$

where  $L(k)$  are the Fourier components of (66) which can be easily evaluated using the definition (33) (keeping in mind all the conventions from Appendix A) to give

$$L(k) = \frac{1}{e^{\frac{|k|}{g}} - 1} \quad (73)$$

Now using the formula

$$\int_0^\infty \frac{dk}{k} \frac{e^{-ikz}}{e^k - 1} = C_1 + izC_2 + \log \Gamma(1 + iz) \quad (74)$$

we see that our expression reduces exactly to the Dorey-Hofman-Maldacena integral representation:

$$-i \oint_{|x|=1} \frac{dx}{2\pi} \oint_{|y|=1} \frac{dy}{2\pi} \frac{1}{x_4^+ - x} \frac{1}{\tilde{x}_4^+ - y} \log \Gamma \left( 1 + ig \left( x + \frac{1}{x} - y - \frac{1}{y} \right) \right) \quad (75)$$

## 5 Conclusions

In this paper we made an observation that for a wide range of relativistic integrable field theories, the overall scalar factor of the S-matrix which makes the S-matrix crossing invariant can be expressed as a convolution of simple kernels appearing in the nested Bethe ansatz. In this way the complicated structure of this scalar factor just comes about from convolutions of simple ingredients. This must be a reflection of some, yet to be discovered, structural self-consistency of nested Bethe ansatz and crossing symmetry.

We argue that if we assume such property to be general, we may reconstruct, up to a discrete set of choices, the extremely complicated BHL/BES dressing factor just starting from the asymptotic Bethe ansatz of [12]. The structure of the dressing phase appearing as a double convolution of various kernels is a direct counterpart of similar formulas appearing in completely conventional relativistic integrable field theories.

Let us comment on the present observation in relation to various proposals which have been made concerning the understanding of the AdS/CFT dressing phase. It has been suggested that the immensely complicated structure of the dressing factor suggests that it arises from some hidden levels of some more fundamental Bethe ansatz ([23], see also [24, 25, 26]). Alternatively, that it could arise from scattering around some filled new vacuum state<sup>6</sup> [27]. The main point that we wanted to make in the present paper is that the complicated structure of the dressing factor is in fact very natural and not in any way more complicated than for the ordinary relativistic  $O(4)$  model. The difference lies only in the different expressions for the kernels appearing in the nested Bethe ansatz - and this leads, almost accidentally, to more complicated formulas. So in a way the fact that the dressing kernel appears to be tightly linked to the nested structure is not related to some ‘stringy’ phenomenae specific for AdS/CFT but is rather a generic fact present also in quite simple relativistic integrable field theories. We must emphasize, however, that this discussion does not preclude the existence of different formulations of the AdS theories in line of [24, 25] related to various ways of quantizing the string using e.g. lightcone or covariant methods.

Finally we would like to emphasize that there remains a lot to be understood. Foremost is why the observation of the present paper seems to work at

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<sup>6</sup>Although such interpretation is problematic for short operators/lengths.

all. Furthermore, in our construction there is a set of discrete choices<sup>7</sup> which gives the correct formula. At the present moment we do not understand what physical/mathematical principle picks the correct choice. We hope that the precise understanding of these issues will shed new light on the structure of integrable field theories in general and on the rather mysterious features of the AdS dressing factor.

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## A Fourier transform conventions

There are a few different conventions for the definitions of the convolution and the Fourier transform. To make the discussion more clear and help the reader follow the computations we present our conventions and some basic relations used in the whole paper.

- Convolution

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x-y)g(y)dy \quad (76)$$

- Fourier transform

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}dx \quad (77)$$

- Inverse Fourier transform

$$\check{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k)e^{-ikx}dk \quad (78)$$

- Fourier transform of the convolution

$$\widehat{(f * g)}(k) = \hat{f}(k)\hat{g}(k) \quad (79)$$

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<sup>7</sup>Both for the AdS case and for relativistic theories.

- Fourier transform of the inverse of the function  $((f * f^{-1})(x) = \delta(x))$

$$\widehat{f^{-1}}(k) = \frac{1}{\hat{f}(k)} \quad (80)$$

- Fourier transform of the kernel  $\phi_a$  defined in (33)

$$\widehat{\phi}_a(k) = -(\text{sgn } a) e^{-|ak|} \quad (81)$$

## B Proof of (50)

To find the formula for the dressing factor we have to solve equations (44)–(47). The Fourier transforms of these equations take the following form

$$\begin{cases} \hat{\rho}_1 + \hat{\rho}_1^h = \hat{\phi}_{-\frac{1}{2}} \hat{\rho}_3 + \hat{\phi}_1 \hat{\rho}_1, \\ \hat{\rho}_2 + \hat{\rho}_2^h = \hat{\phi}_{-\frac{1}{2}} \hat{\rho}_3 + \hat{\phi}_1 \hat{\rho}_2, \\ \hat{\rho}_3 + \hat{\rho}_3^h = \hat{\phi}_{-\frac{1}{2}} \hat{\rho}_1 + \hat{\phi}_{-\frac{1}{2}} \hat{\rho}_2 + \hat{\phi}_{-\frac{1}{2}} \hat{\rho}_4 + \hat{\phi}_1 \hat{\rho}_3, \\ \hat{\rho}_k + \hat{\rho}_k^h = \hat{\phi}_{-\frac{1}{2}} \hat{\rho}_{k+1} + \hat{\phi}_{-\frac{1}{2}} \hat{\rho}_{k-1} + \hat{\phi}_1 \hat{\rho}_k, \end{cases} \quad \text{for } 3 < k \leq n \quad (82)$$

Our claim is that the solution of (82) is of the form

$$\hat{\rho}_{k-1} = \hat{\phi}_{-\frac{1}{2}} \frac{1 + (\hat{\phi}_{-1})^{k-3}}{1 + (\hat{\phi}_{-1})^{k-2}} \hat{\rho}_k + f_k(\hat{\rho}_1^h, \dots, \hat{\rho}_{k-1}^h), \quad \text{for } 3 \leq k \leq n \quad (83)$$

where  $f_k$  for each  $k = 1, \dots, n$  is a function which does not depend on any of  $\hat{\rho}_l$  for  $l = 1, \dots, n+1$ . We will prove this formula by induction.

A straightforward computation shows that our claim is true for  $k = 3$ . Let us assume that (83) holds for a given  $k$ . Then solving

$$\begin{cases} \hat{\rho}_k + \hat{\rho}_k^h = \hat{\phi}_{-\frac{1}{2}} \hat{\rho}_{k+1} + \hat{\phi}_{-\frac{1}{2}} \hat{\rho}_{k-1} + \hat{\phi}_1 \hat{\rho}_k \\ \hat{\rho}_{k-1} = \hat{\phi}_{-\frac{1}{2}} \frac{1 + (\hat{\phi}_{-1})^{k-3}}{1 + (\hat{\phi}_{-1})^{k-2}} \hat{\rho}_k + f_k(\hat{\rho}_1^h, \dots, \hat{\rho}_{k-1}^h) \end{cases} \quad (84)$$

we get (83) taking into account the simple equality  $\hat{\phi}_{-1} = (\hat{\phi}_{-\frac{1}{2}})^2$ . Now we can plug it into the equation for the dressing factor (49):

$$\phi_\sigma = -\frac{1}{2} \phi_{\frac{1}{2}} * \Phi \quad (85)$$

and get

$$\hat{\phi}_\sigma = \frac{1}{2} \hat{\phi}_{-1} \frac{1 + (\hat{\phi}_{-1})^{n-2}}{1 + (\hat{\phi}_{-1})^{n-1}}, \quad \text{for } n \geq 2 \quad (86)$$

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